

Relativity theory of clocks and rulers

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Abstract

Special Relativity (SR) kinematics is derived from very intuitive assumptions. Contrary to standard Einstein's derivation, no light signal is used in the construction nor it is assumed to exist. Instead we postulate the existence of two equivalence classes of physical objects: proportional clocks and proportional rulers. Simple considerations lead to Lorentz kinematics as one of three generic cases. The Lorentz case is characterized by the maximal relative speed of physical objects. The two others are the Galilean and the Euclidean cases.

1. Preliminaries

Writing this paper was inspired by the recent interest to so called Double Special Relativity (DSR) theories [1]. Their authors speculate on the possible modification of standard dispersion relation and on the fundamental frequency dependence of the velocity of light. The present paper is not affirmative or critical to those theories. Our aim is to show that SR can be derived from the set of intuitive assumptions which does not ascribe a special role to light signals. This result seems to be interesting by itself. Only as a byproduct we can point out that consequently the DSR physics of c can be studied without basic SR restrictions.

Let us specify several elementary, intuitive notions.

NOTION 1: **World manifold and world lines**

We assume that the histories of physical objects can be represented by lines in an abstract manifold. Properties of this manifold are to be derived. For simplicity we assume at the moment that the manifold is of dimension 2 and locally can be mapped to \mathbb{R}^2 .

NOTION 2: **Assembling at relative rest**

We introduce this intuitive notion. In the world manifold it is represented by overlapping of two world lines.

ASSUMPTION 1: **Class of proportional clocks and rulers**

We assume existence of the class of proportional *clocks* whose counts rates are constant along their world lines when they assemble at rest. We also assume existence of the class of proportional *rulers* whose world lines of graduation points are such correlated, that when one pair assembles at rest then also the other pairs assemble at rest. We assume that the members of the classes stay proportional each time they assemble at rest. (**Homogeneity 1**) If two clocks/rulers are proportional to the third, they are proportional to each other (class of equivalence).

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NOTION 3: Time intervals and lengths

When clocks scale is fixed (by choosing a reference clock), proportional clocks measure local time interval along their world lines. Correspondingly, proportional rulers measure (locally) the lengths of objects staying at rest with the ruler.

DEFINITION 1: Blent velocity

The notions already collected allows us to define a special kind of relative velocity: defined for clocks passing rulers. If a clock passes the length L (length of the ruler) in the time interval Δt (time of the clock) we define the blent velocity of the clock relatively to the ruler as

$$u = L/\Delta t. \tag{1}$$

Choosing orientation of the ruler, we can define positive and negative velocities.

ASSUMPTION 2: Relativity 1

Take two sets, each containing one ruler and one clock assembled at rest. If the systems pass each other, the values of blent velocities of these systems are equal.

Our aim will be to construct the algebra of velocities. Here we will need the last but not least assumption:

ASSUMPTION 3: Relativity 2

Locally the composition of blent velocities depends only on the relative states of motions.

The last assumption means that having three bodies meeting somewhere, it is enough to know two relative blent velocities to be able to calculate the third one.

The Assumption 3 was formulated locally. It means that we have restricted ourselves to only such cases that fulfill the following assumption:

ASSUMPTION 4: Locality

We consider only pairs of such small rulers for which blent velocities of separate fragments are equal to each other. It is equivalent to the postulate of inertial frames in the standard derivation. Instead of locality we can consider only special states of relative motions.

Equipped with these notions and assumptions, we can sketch the key picture of our derivation.

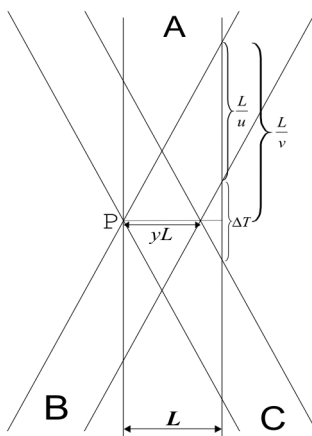


Fig. 1

2. Key picture

Take three equal rulers **A**, **B** and **C**. Let **B** pass **A** with blent velocity u (to the right) and let **C** pass **A** with blent velocity $-u$ (to the left). Let left ends of the rulers meet at some point **P**. Then we have three generic cases concerning their right ends.

CASE I: (Lorentz case - shown at the picture)

Right end of **A** meets first the right end of **C** and then meets the right end of **B**

CASE II: (Galilean case)

The three right ends meet at one point.

CASE III: (Euclidean case)

The order is opposite to case I.

Thus the cases distinguish by the time interval ΔT (measured by the clock fixed at the right end of **A**) between right ends meetings of **A-C** and **A-B**. Choosing a convention we can say that this interval is positive in the case I and equals zero or is negative in the remaining two cases respectively.

CONCLUSION 1: Synchronization of clocks

Observe that the described construction can serve as a procedure of clock synchronization. In fact, having clocks fixed at the ends of **A**, we can synchronize them in such a way that they show the same time at **P** and at half time interval of ΔT .

This synchronization works in all three cases! However, we restrict our considerations here only to the Case I.

Having synchronized clocks, we can define **standard velocity** v in a natural way. It is easy to perceive, that for our picture, eg. for the velocities v and u of **B** relatively to **A**, we have the relation

$$\frac{1}{v} = \frac{1}{u} + \frac{\Delta T}{2L} \quad (2)$$

where L is the length of rulers.

CONCLUSION 2: Lorentz contraction

It is also worth to observe, that once the clocks of **A** are synchronized, we can define the length of moving body: the length ascribed to it as a distance between its ends at world points regarded as simultaneous in the clock synchronization of **A**. Thus **in the synchronized reference system of A** we can ascribe the length L' to the ruler of the length L moving with velocity v :

$$L' = yL \quad (3)$$

where y is a contraction factor.

We will show that our former assumptions are sufficient to derive the dependence $y(v)$. (Independence of L follows from assumption 4.)

The contracted length of **B** (or **C**) is the distance between point **P** and the point of **B** and **C** right ends meeting. So we get

$$yL = \frac{v}{u} \quad (4)$$

In order to derive the dependence $y(v)$ it is convenient to consider a series of subsequent "additions" of equal velocities.

3. Sequence of boosts

Let us consider a question how the velocity and the contraction of C with respect to B depend on y and v .

Consider a "boosted" picture.

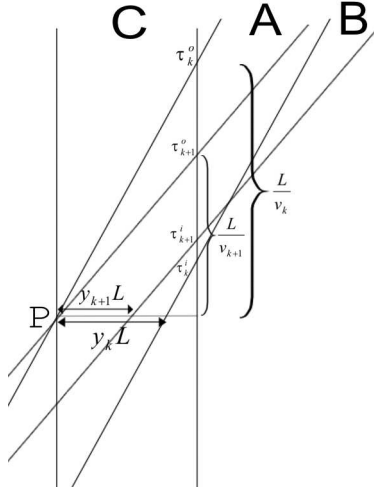


Fig. 2

We have added indexes k for quantities describing the "old" motion with blent velocity u and indexes $k+1$ for quantities describing the "new" motion of C relatively to B with the "doubled" relative velocity.

Going back to figure 1 we can examine that there hold the following relations between time intervals:

$$v_k(\tau_k^o - \tau_k^i) = y_k L \quad (5)$$

$$\frac{\tau_k^o - \tau_{k+1}^i}{\tau_k^o - \tau_k^i} = \frac{y_k L}{L} = y_k. \quad (6)$$

The symmetry between the "forward" and "backward" motion at figure 1 implies that:

$$\tau_k^o - \tau_{k+1}^o = \tau_{k+1}^i - \tau_{k+1}^i. \quad (7)$$

Combining these relations we get

$$\tau_k^o - \tau_{k+1}^o = \frac{1}{2} \frac{L}{v_k} y_k^2. \quad (8)$$

If clocks of C are synchronized in such a way that C-time of P is equal 0 then:

$$\tau_k^o = \frac{L}{v_k}, \quad \tau_{k+1}^o = \frac{L}{v_{k+1}} \quad (9)$$

and we get

$$\frac{1}{v_{k+1}} = \frac{1}{v_k} \left(1 - \frac{1}{2} y_k^2\right). \quad (10)$$

We also get that

$$y_{k+1} = \frac{y_k^2}{2 - y_k^2}. \quad (11)$$

Then we can use obtained v_{k+1} and y_{k+1} to construct next step of iteration and so on.

4. Results

The series given by (11) decrease to zero if started from contraction y smaller than one:

$$\lim_{k \rightarrow \infty} y_k = 0. \quad (12)$$

(It can be shown, when we observe that the series (11) is restricted from above by the series $y_{k+1} = y_k^2$.)

We goes to the following conclusions:

CONCLUSION 3: Maximal speed

By direct insertion one can verify that there holds the relation

$$\frac{v_k^2}{1 - y_k^2} = \frac{v_{k+1}^2}{1 - y_{k+1}^2} = \text{const} = c^2. \quad (13)$$

The new introduced constant c is the maximal relative velocity which we can obtain combining relative motions (what is clear from (13) and from (12)).

The relation (13) was obtained for the series (11). But it is easy to extend it to all relative motions obtained from "adding" and "subtracting" any motions from the series and to whichever motion obtained from them.

CONCLUSION 4: Key result: Lorentz contraction factor

Thus we can generally rewrite (13) in the form:

$$y = \sqrt{1 - \frac{v^2}{c^2}}. \quad (14)$$

This is our desired relation between the velocity and the contraction factor.

Having established relation (14) it is straightforward to derive standard Lorentz group of transformations between moving systems of rulers and clocks and the Lorentz algebra of velocities (repeating in fact the Lorentz path). It restores all local Minkowski space structure.

5. Conclusions

We have shown in the simple two dimensional case that the very natural physical assumptions on the existence of proportional clocks and proportional rulers, together with homogeneity and relativity assumptions, allows us to rediscover local Minkowski structure of space-time. The structure of Minkowski space can be derived as one of three generic cases (containing also Galilean and Euclidean case). The signals of light had no special role in our derivation. In fact the existence of light was even not mentioned. The parameter of maximal velocity appears as a consequence of Lorentz contraction. The fact of contraction is treated as a physical phenomenon distinguishing Minkowski case from the two other generic cases. No special form of Lorentz contraction was assumed. It was derived from fundamental assumptions and the maximal speed appears in the derivation as the only free parameter whose concrete value has to be fixed from experiment. The above derivation opens some room for speculations on a nontrivial physics of the speed of light. In fact, we have not assumed that our maximal speed is the speed of light. This conclusion - if valid - has to be derived from physical considerations (eg. from considerations of the speed of interactions that makes our rulers and clocks really proportional). But such identification in principle can be also rejected.

References

- [1] G. Amelino-Camelia, gr-qc/0210063 and references given therein.